Exercise 74

If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

Solution

Use the product rule and the chain rule to differentiate $F(x)$.

$$\begin{aligned}
F'(x) &= \frac{d}{dx}[F(x)] \\
&= \frac{d}{dx}[f(xf(xf(x)))] \\
&= f'(xf(xf(x))) \cdot \frac{d}{dx}[xf(xf(x))] \\
&= f'(xf(xf(x))) \cdot \left\{ \left[\frac{d}{dx}(x)\right] f(xf(x)) + x \left[\frac{d}{dx} f(xf(x))\right] \right\} \\
&= f'(xf(xf(x))) \cdot \left\{ (1) f(xf(x)) + x \left[f'(xf(x)) \cdot \frac{d}{dx}[xf(x)] \right] \right\} \\
&= f'(xf(xf(x))) \cdot \left\{ f(xf(x)) + x \left[f'(xf(x)) \cdot [f(x) + xf'(x)] \right] \right\}
\end{aligned}$$

Evaluate it at $x = 1$.

$$\begin{aligned}
F'(1) &= f'(f(f(1))) \cdot \left\{ f(f(1)) + \left[f'(f(1)) \cdot [f(1) + f'(1)] \right] \right\} \\
&= f'(f(2)) \cdot \left\{ f(2) + \left[f'(2) \cdot (2 + 4) \right] \right\} \\
&= f'(3) \cdot \{3 + [5 \cdot (6)]\} \\
&= (6) \cdot [3 + (30)] \\
&= (6) \cdot (33) \\
&= 198
\end{aligned}$$